# CSCI 7000 Fall 2023: Inclusion–Exclusion

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1. Generatingfunctionology Chapter 4 Exercise 9 (p. 159), reproduced verbatim here:

   > Let $G$ be a graph of $n$ vertices, and let positive integers $x, \lambda$ be given. Let $P(\lambda; x; G)$ denote the number of ways of assigning one of $\lambda$ given colors to each of the vertices of $G$ in such a way that exactly $x$ edges of $G$ have both endpoints of the same color.
   >
   > Formulate the question of determining $P$ as a sieve [inclusion-exclusion] problem with a suitable set of objects and properties. Find a formula for $P(\lambda; x; G)$, and observe that it is a polynomial in the two variables $\lambda$ and $x$. The *chromatic polynomial* of $G$ is $P(\lambda; 0; G)$.

2. Given non-negative integers $k, n, d$, find the number of non-negative integer solutions to the equation

   $$x_1 + x_2 + \cdots + x_k = n$$

   such that all $x_i$ satisfy $0 \leq x_i \leq d$.

3. (Stanley, *Enumerative Combinatorics*, Volume I, second edition, Chapter 2, Exercise 14). Let $A_k(n)$ denote the number of collections $S$ of $k$ subsets of $\{1, \ldots, n\}$ such that no element of $S$ is a subset of another element of $S$. Show that $A_1(n) = 2^n$ and $A_2(n) = (1/2)(4^n - 2 \cdot 3^n + 2^n)$. Try to compute $A_k(n)$ for $k = 3, 4$. Can you see the pattern? See for how large a $k$ can you get a general formula (as a function of $n$).

## Resources

- van Lint & Wilson Chapter 10
- Generatingfunctionology Section 4.2 for a generating function view of inclusion–exclusion
- Generatingfunctionology p. 113 for average number of fixed points of a permutation via inclusion–exclusion
- Enumerative Combinatorics Chapter 2